4201. There are $6^3 = 216$ outcomes in the possibility space. Consider the outcomes in which the first two $X+Y$ add up to the third Z, classified by the value of Z:

Z	(X, Y)
1	None
2	$(1,1)$
3	$(1,2),(2,1)$
4	$(1,3),(2,2),(3,1)$
5	$(1,4),(2,3),(3,2),(4,1)$
6	$(1,5),(2,4),(3,3),(4,2),(5,1)$

There are 15 outcomes with Z largest. There are also 15 with X largest and 15 with Y largest. So, the probability is $^{45}/_{216}$. This is$^5/_{24}$, as required.

4202. The normal to the curve at (p, p^2) has equation

$$\begin{aligned} & y - p^2 = -\tfrac{1}{2p}(x-p) \\ \Longrightarrow\ & y = -\tfrac{1}{2p}x + \tfrac{1}{2} + p^2. \end{aligned}$$

Solving this simultaneously with $y = x^2$,

$$\begin{aligned} & x^2 = -\tfrac{1}{2p}x + \tfrac{1}{2} + p^2 \\ \Longrightarrow\ & x^2 + \tfrac{1}{2p}x - \tfrac{1}{2} - p^2 = 0 \\ \Longrightarrow\ & 2px^2 + x - p - 2p^3 = 0 \\ \Longrightarrow\ & x = p, -p - \tfrac{1}{2p}. \end{aligned}$$

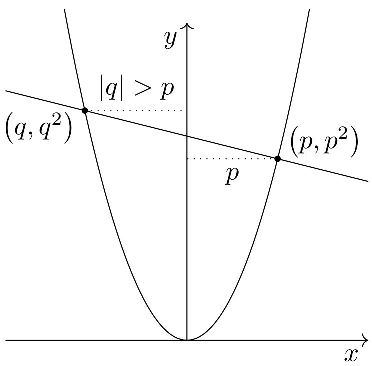
So, the x component of the length of the chord is

$$\begin{aligned} s_x &= p - \left(-p - \tfrac{1}{2p}\right) \\ &= 2p + \tfrac{1}{2p}. \end{aligned}$$

Since both terms are positive, $s_x > 2p$. So, the length of the chord is greater than $2p$, as required.

— Alternative Method —

Consider the symmetry of the scenario. The chord is normal to the curve. And the gradient of the curve is positive in the positive quadrant. So, the gradient of the chord is negative, meaning that the y coordinate of the other endpoint is greater than p^2. Call it q^2.

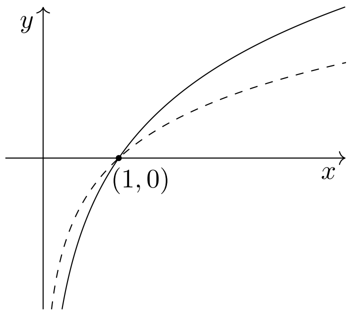


We know that $|q| > p$, as shown in the diagram. So, $p - q > 2p$, as required.

4203. (a) The least possible value of $\mathrm{f}(x) + \mathrm{g}(x)$ is $a+b$, and the greatest possible value is $c+d$. Any values in between these two may be attainable. So, the smallest set which can be guaranteed to contain the range is the interval $[a+b, c+d]$.

(b) The least possible value of $\mathrm{f}(x) - \mathrm{g}(x)$ is $a-d$ and the greatest is $c-b$. Again, any values in between these two may be attainable. So, the smallest set which can be guaranteed to contain the range is the interval $[a-d, c-b]$.

4204. (a) The logarithm curves are reflections of $y = 2^x$ and $y = 3^x$ in the line $y = x$. They cross at $(1, 0)$. The curve $y = \log_3 x$ is shown dashed:



(b) Raising base and input of $y = \log_a x$ to the power $\log_a b$ gives

$$\begin{aligned} & y = \log_b x^{\log_a b} \\ \Longrightarrow\ & y = \log_a b \times \log_b x. \end{aligned}$$

Hence, $y = \log_a x$ is a stretch, by scale factor $\log_a b$ in the y direction, of $y = \log_b x$.

4205. For parts, let $u = 3x - 6$ and $\frac{dv}{dx} = (2x+3)^{-\frac{1}{2}}$. This gives $\frac{du}{dx} = 3$ and $v = (2x+3)^{\frac{1}{2}}$. Substituting into the parts formula,

$$\begin{aligned} & \int \frac{3x-6}{\sqrt{2x+3}}\,dx \\ &= (3x-6)(2x+3)^{\frac{1}{2}} - \int 3(2x+3)^{\frac{1}{2}}\,dx \\ &= (3x-6)(2x+3)^{\frac{1}{2}} - (2x+3)^{\frac{3}{2}} + c \\ &\equiv \big(3x-6-(2x+3)\big)(2x+3)^{\frac{1}{2}} + c \\ &\equiv (x-9)\sqrt{2x+3} + c, \text{ as required.} \end{aligned}$$

— Alternative Method —

Let $u = 2x+3$, so that $\frac{1}{2}\,du = dx$ and $x = \frac{1}{2}(u-3)$. Enacting the substitution,

$$\begin{aligned} & \int \frac{3x-6}{\sqrt{2x+3}}\,dx \\ &= \int \frac{\frac{3}{2}(u-3)-6}{\sqrt{u}} \cdot \tfrac{1}{2}\,du \\ &= \int \tfrac{3}{4}u^{\frac{1}{2}} - \tfrac{21}{4}u^{-\frac{1}{2}}\,du. \end{aligned}$$

Integrating term by term, this is

$$\begin{aligned}&\tfrac{1}{2}u^{\frac{3}{2}}-\tfrac{21}{2}u^{\frac{1}{2}}+c\\ \equiv\ &\left(\tfrac{1}{2}u-\tfrac{21}{2}\right)u^{\frac{1}{2}}+c\\ =\ &\left(\tfrac{1}{2}(2x+3)-\tfrac{21}{2}\right)(2x+3)^{\frac{1}{2}}+c\\ \equiv\ &(x-9)\sqrt{2x+3}+c,\ \text{as required.}\end{aligned}$$

4206. (a) The gradient at (p, e^p) is e^p. So, $\tan\theta = e^p$. This gives

$$\begin{aligned}&\tan^2\theta = e^{2p}\\ \Longrightarrow\ &\cot^2\theta = e^{-2p}\\ \Longrightarrow\ &1+\cot^2\theta = 1+e^{-2p}\\ \Longrightarrow\ &\operatorname{cosec}^2\theta = 1+e^{-2p}\\ \Longrightarrow\ &\sin^2\theta = \frac{1}{1+e^{-2p}}\\ &\equiv \frac{e^{2p}}{e^{2p}+1},\ \text{as required.}\end{aligned}$$

4207. Assume, for a contradiction, that $\sqrt[3]{2}$ is rational, so can be written as p/q, where $p, q \in \mathbb{N}$. Let p have m factors of 2, and q have n factors of 2.

$$\begin{aligned}&\sqrt[3]{2} = \frac{p}{q}\\ \Longrightarrow\ &2 = \frac{p^3}{q^3}\\ \Longrightarrow\ &2q^3 = p^3.\end{aligned}$$

Consider the number of factors of 2 on each side of the above equation. Setting this up as an equation:

$$1+3m = 3n.$$

The RHS is a multiple of 3, but the LHS isn't. This is a contradiction. Hence, $\sqrt[3]{2}$ is irrational. □

——— Alternative Method ———

Assume, for a contradiction, that $\sqrt[3]{2}$ is rational, so can be written as $\frac{p}{q}$, where p and q are integers with $\text{hcf}(p, q) = 1$.

$$\begin{aligned}&\sqrt[3]{2} = \frac{p}{q}\\ \Longrightarrow\ &2 = \frac{p^3}{q^3}\\ \Longrightarrow\ &2q^3 = p^3.\end{aligned}$$

The LHS is even, so the RHS is even. This implies that p is even. Write it as $p = 2k$. Substituting in,

$$\begin{aligned}&2q^3 = (2k)^3 = 8k^3\\ \Longrightarrow\ &q^3 = 4k^3.\end{aligned}$$

Since the RHS is even, q must be even. So, p and q have a common factor of 2. This contradicts $\text{hcf}(p, q) = 1$. Hence, $\sqrt[3]{2}$ is irrational. □

4208. (a) We are given that $a \propto v^2$, so $a = kv^2$ for some constant k. As a DE in v and t, this is

$$\begin{aligned}&\frac{dv}{dt} = kv^2\\ \Longrightarrow\ &\frac{1}{v^2}\frac{dv}{dt} = k\\ \Longrightarrow\ &\int \frac{1}{v^2}\frac{dv}{dt}\,dt = \int k\,dt.\end{aligned}$$

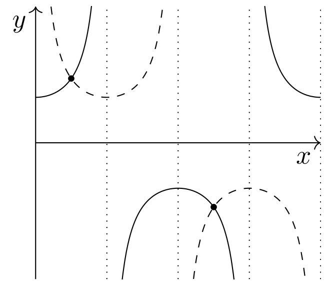
(b) The LHS of the above simplifies according to the (integral) chain rule, giving

$$\begin{aligned}&\int \frac{1}{v^2}\,dv = \int k\,dt\\ \Longrightarrow\ &-v^{-1} = kt - c\\ \Longrightarrow\ &v^{-1} = c - kt\\ \Longrightarrow\ &v = \frac{1}{c-kt},\ \text{as required.}\end{aligned}$$

4209. The boundary equation is $\sec x = \operatorname{cosec} x$. Taking the reciprocal of both sides, this is

$$\begin{aligned}&\sin x = \cos x\\ \Longrightarrow\ &\tan x = 1\\ \Longrightarrow\ &x = \tfrac{\pi}{4} + n\pi.\end{aligned}$$

The functions sec and cosec also have sign changes at asymptotes. So, we sketch the graphs $y = \sec x$ and $y = \operatorname{cosec} x$ (dashed):



The intersections marked are at $\pi/4$ and $5\pi/4$, and the vertical asymptotes are at multiples of $\pi/2$. We need the x values at which both curves are well defined, and the dashed curve is above the solid curve. This is

$$x \in \left(0, \tfrac{\pi}{4}\right) \cup \left(\tfrac{\pi}{2}, \pi\right) \cup \left(\tfrac{5\pi}{4}, \tfrac{3\pi}{2}\right).$$

4210. For intersections,

$$\begin{aligned}&4x^2 - 2 = x^4 + k\\ \Longrightarrow\ &x^4 - 4x^2 + k + 2 = 0.\end{aligned}$$

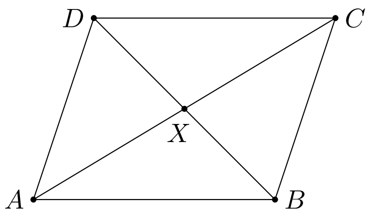
This is a quadratic in x^2. For the graphs to be tangent, we require a double root. This can either occur if $x^2 = 0$ is a root of the quadratic, or if the quadratic has discriminant $\Delta = 0$.

① For $x^2 = 0$, we require the biquadratic to have no constant term, so $k = -2$. In this case, $x^2(x^2 - 4) = 0$; the curves are tangent at $x = 0$.

② For $\Delta = 0$, we require $16 - 4(k - 2) = 0$, so $k = 2$. In this case, $(x^2 - 2)^2 = 0$; the curves are tangent at $\pm\sqrt{2}$.

So, $k = \pm 2$.

4211. Call the central point X.



Since X is the midpoint of the diagonals, we know that $\overrightarrow{AX} = \overrightarrow{XC}$ and $\overrightarrow{DX} = \overrightarrow{XB}$. Therefore,

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AX} + \overrightarrow{XB} \\ &= \overrightarrow{XC} + \overrightarrow{DX} \\ &= \overrightarrow{DC}.\end{aligned}$$

Hence, $ABCD$ is a parallelogram. QED.

4212. (a) $\mathbb{P}(A \mid B)$ cannot be calculated.

(b) $\mathbb{P}(B \mid A) = \frac{0.1}{0.2} = 0.5$.

(c) $\mathbb{P}(A \cap C)$ cannot be calculated.

(d) $\mathbb{P}(B \cap C \mid A) = \frac{0.05}{0.2} = 0.5$.

4213. The function is a positive octic, so its range is of the form $[a, \infty)$. The minimum value occurs at the minimum of the quadratic $1 + x + x^2$. Completing the square,

$$x^2 + x + 1 \equiv \left(x + \tfrac{1}{2}\right)^2 + \tfrac{3}{4}.$$

Hence, the minimum value of the octic is $(3/4)^4$. So, its range is $\left[81/256, \infty\right)$.

4214. Since $\theta \in [0, \pi/36]$ and is therefore small, we can use a small-angle approximation: $\cos\theta \approx 1 - \frac{1}{2}\theta^2$. Also, the binomial expansion, neglecting terms higher than θ^2, gives

$$\left(1 + \theta^2\right)^{-1} = 1 - \theta^2 + \dots.$$

Combining these,

$$\begin{aligned}f(\theta) &\approx \left(1 - \tfrac{1}{2}\theta^2\right)\left(1 - \theta^2\right) \\ &\approx 1 - \tfrac{3}{2}\theta^2.\end{aligned}$$

Approximating the integral,

$$\begin{aligned}&\int_0^{\frac{\pi}{36}} \frac{\cos\theta}{1 + \theta^2}\, d\theta \\ &\approx \int_0^{\frac{\pi}{36}} 1 - \tfrac{3}{2}\theta^2 \, d\theta \\ &= \left[\theta - \tfrac{1}{2}\theta^3\right]_0^{\frac{\pi}{36}} \\ &= 0.0869 \text{ (3sf)}.\end{aligned}$$

4215. Differentiating implicitly,

$$\begin{aligned}&(x - k)(y - k) + x^2y^2 = 0 \\ \Longrightarrow\ &(y - k) + (x - k)\tfrac{dy}{dx} + 2xy^2 + 2x^2y\tfrac{dy}{dx} = 0.\end{aligned}$$

So, for SPs, $y - k + 2xy^2 = 0$. Rearranging this,

$$x = \frac{k - y}{2y^2}.$$

Substituting into the original equation,

$$\left(\frac{k - y}{2y^2} - k\right)(y - k) + \left(\frac{k - y}{2y^2}\right)^2 y^2 = 0.$$

Either $y = k$, or

$$\begin{aligned}&\left(\frac{k - y}{2y^2} - k\right) - \frac{k - y}{4y^2} = 0 \\ \Longrightarrow\ &2k - 2y - 4ky^2 - (k - y) = 0 \\ \Longrightarrow\ &4ky^2 + y - k = 0 \\ \Longrightarrow\ &y = \frac{-1 \pm \sqrt{1 + 16k^2}}{8k}.\end{aligned}$$

When k is large and positive,

$$y \approx \frac{\pm\sqrt{16k^2}}{8k} = \pm\tfrac{1}{2}, \text{ as required.}$$

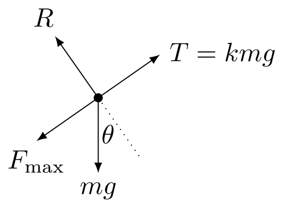
4216. The derivative of $a\,f(x) + b$ is $a\,f'(x)$. The top of the fraction is $f'(x)$, which is a multiple of this, so we can integrate by the reverse chain rule (integration by inspection):

$$\begin{aligned}&\int \frac{f'(x)}{(a\,f(x) + b)^2}\, dx \\ &= \frac{1}{a}\int \frac{a\,f'(x)}{(a\,f(x) + b)^2}\, dx \\ &= \frac{1}{a} \cdot -\frac{1}{(a\,f(x) + b)} + c \\ &\equiv -\frac{1}{a(a\,f(x) + b)} + c.\end{aligned}$$

— Nota Bene —

As with all integration by inspection, the logic of the above is best understood by performing the relevant calculations in reverse: differentiating by the chain rule.

4217. Assuming that the string is light, the tension in the vertical section of string is $T = kmg$. Assuming that the pulley is smooth, the tension throughout the string is $T = kmg$. For the boundary case in which the block is on the point of sliding up the slope, the forces for the block on the slope are



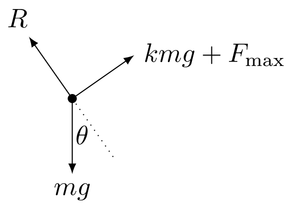
Resolving perpendicular to the slope,

$$R = mg\cos\theta = \tfrac{4}{5}mg.$$

So, $F_{\max} = \mu R = \frac{1}{5}mg$. Parallel to the slope,

$$\begin{aligned} & kmg - \tfrac{1}{5}mg - mg\sin\theta = 0 \\ \Longrightarrow\ & k = \tfrac{4}{5}. \end{aligned}$$

Assuming instead that the block is on the point of sliding down the slope, the force diagram is



Resolving parallel to the slope,

$$\begin{aligned} & kmg + \tfrac{1}{5}mg - mg\sin\theta = 0 \\ \Longrightarrow\ & k = \tfrac{2}{5}. \end{aligned}$$

So, the ratio of masses satisfies $k \in \left[2/5, 4/5\right]$.

4218. We can take the index i out of the j-indexed sum, as it is a constant as far as j is concerned:

$$\begin{aligned} S &= \sum_{i=1}^{n}\left(\sum_{j=1}^{n} ij\right) \\ &\equiv \sum_{i=1}^{n}\left(i\sum_{j=1}^{n} j\right). \end{aligned}$$

We sum over j, using the standard result for the sum of the first n integers. The sum $\frac{1}{2}n(n+1)$ is then a common factor in the i-indexed sum:

$$\begin{aligned} S &= \sum_{i=1}^{n}\left(i\cdot\tfrac{1}{2}n(n+1)\right) \\ &\equiv \tfrac{1}{2}n(n+1)\sum_{i=1}^{n} i \\ &\equiv \left(\tfrac{1}{2}n(n+1)\right)\left(\tfrac{1}{2}n(n+1)\right) \\ &\equiv \tfrac{1}{4}n^2(n+1)^2, \text{ as required.} \end{aligned}$$

4219. Using log rules, we rewrite the first equation:

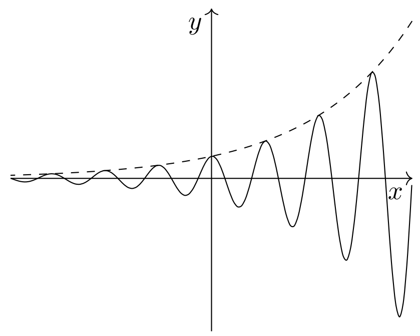
$$\begin{aligned} & \log_2 x + 2\log_4 y = 1 \\ \Longrightarrow\ & \log_2 x + \log_4 y^2 = 1 \\ \Longrightarrow\ & \log_2 x + \log_2 y = 1 \\ \Longrightarrow\ & \log_2(xy) = 1 \\ \Longrightarrow\ & xy = 2. \end{aligned}$$

Solving this with $x + y = 3$ gives $(1,2)$ or $(2,1)$.

4220. The equation for intersections is

$$\begin{aligned} & e^x - e^x\cos x = 0 \\ \Longrightarrow\ & e^x(1-\cos x) = 0. \end{aligned}$$

The latter factor has infinitely many roots, in AP. Furthermore, since the range of $1-\cos x$ is $[0,2]$, the sign of $e^x(1-\cos x)$ never changes: it is non-negative everywhere. So, the graph $y = e^x$ remains at or above $y = e^x\cos x$ everywhere. Hence, every point of intersection must be a point of tangency. Not to scale, the behaviour is



4221. For each square along the left-hand and top edges, the number of paths leading there is 1. For every other square, the number of paths is the sum of the number of paths to the squares immediately above and to the left of it. The first few numbers are shown:

1	1	1	1	1	1	1	1
1	2	3	4				
1	3	6					
1	4						
1							
1							
1							
1							

As can be seen, these are the rules for generating Pascal's triangle. Hence, the number of ways of reaching the bottom-right corner is ${}^{14}\mathrm{C}_7 = 3432$.

— Alternative Method —

We may assume, without loss of generality, that the rook moves one square at a time. Longer moves do not produce distinct paths. To reach the bottom-right corner, 14 moves are required, which must consist of 7 downward moves and 7 rightward moves. There are ${}^{14}\mathrm{C}_7 = 3432$ orders.

4222. Setting $y = 0$ for the limits of the integral,

$$1.5236x^8 - 2.0767x^3 - 6.8814 = 0.$$

This is not analytically solvable. So, we set up N-R. The iteration is

$$x_{n+1} = x_n - \frac{1.5236x_n^8 - 2.0767x_n^3 - 6.8814}{12.1888x_n^7 - 6.2301x_n^2}.$$

Running this with starting points $x_0 = \pm 5$ gives the x intercepts as $x = 1.28436$ and $x = -1.12543$. So, the signed area is calculated by

$$S = \int_{-1.12543}^{1.28436} 1.5236x^8 - 2.0767x^3 - 6.8814\,dx.$$

The definite integration facility on a calculator gives $S = -15.06...$. So, to 3sf, the area enclosed is 15.1 square units.

4223. Solving algebraically,

$$\begin{aligned} &x^2(2x+k) - 1 = 0 \\ \Longrightarrow\ &2x^3 + kx^2 - 1 = 0. \end{aligned}$$

Consider $y = 2x^3 + kx^2 - 1$. For this cubic to have exactly two real roots, it must have a stationary point on the x axis. For SPs,

$$\begin{aligned} &6x^2 + 2kx = 0 \\ \Longrightarrow\ &x = 0, -\tfrac{k}{3}. \end{aligned}$$

Testing $x = 0$ in the original equation, $1 = 0$, so this is not the double root. Testing $x = -\frac{k}{3}$ in the original equation,

$$\begin{aligned} &\left(-\tfrac{k}{3}\right)^2 - \frac{1}{-\frac{2k}{3} + k} = 0 \\ \Longrightarrow\ &k = 3. \end{aligned}$$

At $k = 3$, the equation has a double root at $x = -1$ and a single root at $x = \frac{1}{2}$.

4224. The iteration is Newton-Raphson for the equation $\mathrm{f}'(x) = 0$. Hence, if it converges to α, then $x = \alpha$ must be a root of $\mathrm{f}'(x) = 0$. So, $\mathrm{f}'(\alpha) = 0$.

4225. (a) Two cubics with the same leading coefficient need not intersect. For instance, $y = x^3$ and $y = x^3 + 1$.

(b) This true for $k \in \mathbb{N}$, but not for $k \in \mathbb{Z}^-$. With $k = -1$, the graph is the standard reciprocal $y = \frac{1}{x}$, which does not intersect the x axis.

4226. Translating into algebra,

$$\begin{aligned} &\int_0^k 4x^{\frac{1}{3}} + 2x^{-\frac{1}{3}}\,dx = 18 \\ \Longrightarrow\ &\left[3x^{\frac{4}{3}} + 3x^{\frac{2}{3}}\right]_0^k = 18 \\ \Longrightarrow\ &3k^{\frac{4}{3}} + 3k^{\frac{2}{3}} - 18 = 0 \\ \Longrightarrow\ &(k^{\frac{2}{3}} + 3)(k^{\frac{2}{3}} - 2) = 0 \\ \Longrightarrow\ &k^{\frac{2}{3}} = -3, 2. \end{aligned}$$

Since $k^{\frac{2}{3}} > 0$, we reject the first root. The second root gives $k = \sqrt{8}$.

4227. (a) Multiplied by $\sqrt[3]{2} - 1$, the denominator is

$$\begin{aligned} &\left(\sqrt[3]{2} + 1\right)\left(\sqrt[3]{2} - 1\right) \\ \equiv\ &2^{\frac{2}{3}} - 1. \end{aligned}$$

The first term is irrational, so the given factor fails to rationalise the denominator.

(b) Multiplying top and bottom by $2^{\frac{2}{3}} - 2^{\frac{1}{3}} + 1$,

$$\begin{aligned} &\frac{1}{\sqrt[3]{2} + 1} \\ =\ &\frac{2^{\frac{2}{3}} - 2^{\frac{1}{3}} + 1}{\left(\sqrt[3]{2} + 1\right)\left(2^{\frac{2}{3}} - 2^{\frac{1}{3}} + 1\right)} \\ =\ &\frac{2^{\frac{2}{3}} - 2^{\frac{1}{3}} + 1}{3}. \end{aligned}$$

The denominator is rationalised.

4228. (a) By the chain rule,

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) \\ &\equiv \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx} \\ &\equiv \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}, \text{ as required.} \end{aligned}$$

(b) Differentiating, $\frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = 1 + 3t^2$. So,

$$\begin{aligned} \frac{dy}{dx} &= \frac{1 + 3t^2}{2t} \\ &\equiv \tfrac{1}{2}t^{-1} + \tfrac{3}{2}t. \end{aligned}$$

Using the formula in part (a),

$$\begin{aligned} \frac{d^2y}{dx^2} &\equiv \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} \\ &= \frac{\frac{d}{dt}\left(\frac{1}{2}t^{-1} + \frac{3}{2}t\right)}{2t} \\ &\equiv \frac{-\frac{1}{2}t^{-2} + \frac{3}{2}}{2t} \\ &\equiv \frac{-\frac{1}{2} + \frac{3}{2}t^2}{2t^3}. \end{aligned}$$

Setting the numerator to zero,

$$-\tfrac{1}{2} + \tfrac{3}{2}t^2 = 0$$
$$\Longrightarrow t^2 = 3.$$

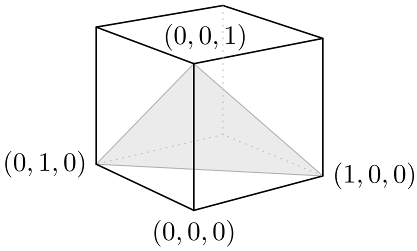
This has single roots at $t = \pm\sqrt{3}$. Also, the denominator does not change sign. So, these are points of inflection. Subbing t values, the coordinates of the points of inflection are

$$\left(\tfrac{1}{3}, \pm\tfrac{4}{3\sqrt{3}}\right).$$

4229. Since $\overrightarrow{AB}$ is parallel to $\mathbf{i} + 3\mathbf{j}$, we know that Δy is three times Δx:

$$-e^k - e^k = 3\big(e^{-k} - e^k\big)$$
$$\Longrightarrow e^k = 3e^{-k}$$
$$\Longrightarrow e^{2k} = 3$$
$$\Longrightarrow k = \tfrac{1}{2}\ln 3$$
$$= \ln\sqrt{3}, \text{ as required.}$$

4230. The possibility space is a cube of side length 1.



The successful region is bounded by the equilateral triangle shown. The region is a triangular-based pyramid. Six such pyramids tessellate to fill the cube. Hence, the probability that the coordinates add to less than 1 is 1/6.

4231. At P, the horizontal and vertical components of velocity are $\sqrt{2}$ ms^{-1}. The height at P is $1.5 - 1.2\cos 45°$. The vertical *suvat* is

$$-(1.5 - 1.2\cos 45°) = \sqrt{2}t - \tfrac{1}{2}gt^2$$
$$\Longrightarrow t = -0.247838, 0.536453.$$

We take the positive value $t = 0.536453$ as the time of flight. At P, the ball has already travelled $1.2 + 1.2\sin 45°$ horizontally. So,

$$d = 1.2 + 1.2\sin 45° + \sqrt{2} \times 0.536453$$
$$= 2.81 \text{ (3sf)}.$$

4232. (a) Using the definition,

$$\mathrm{A}_1(x) + \mathrm{A}_2(x) = 0$$
$$\therefore \quad \frac{x+1}{x-1} + \frac{2x+1}{2x-1} = 0$$
$$\Longrightarrow (x+1)(2x-1) + (2x+1)(x-1) = 0$$
$$\Longrightarrow x = \pm\tfrac{\sqrt{2}}{2}.$$

(b) The definition gives

$$\mathrm{A}_1(x)\,\mathrm{A}_2(x) = \frac{(x+1)(2x+1)}{(x-1)(2x-1)}.$$

In partial fractions, this is

$$1 + \frac{6}{1-2x} - \frac{6}{1-x}$$
$$\equiv 1 + 6(1-2x)^{-1} - 6(1-x)^{-1}.$$

The quadratic binomial expansions are

$$(1-2x)^{-1} = 1 + 2x + 4x^2 + ...,$$
$$(1-x)^{-1} = 1 + x + x^2 + ...$$

So, the overall quadratic approximation is

$$1 + 6\big(1 + 2x + 4x^2\big) - 6\big(1 + x + x^2\big)$$
$$\equiv 1 + 6x + 18x^2.$$

4233. (a) The denominator has a root at $x = 4.5$, so this is the vertical asymptote. Rewriting as a proper algebraic fraction,

$$\frac{6x+1}{2x-9} \equiv \frac{3(2x-9)+28}{2x-9}$$
$$\equiv 3 + \frac{28}{2x-9}.$$

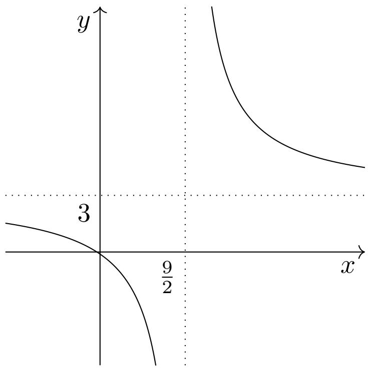
As $x \to \pm\infty$, $y \to 3$. So, there is a horizontal asymptote at $y = 3$.

(b) Using the quotient rule,

$$\frac{dy}{dx} = \frac{6(2x-9) - 2(6x+1)}{(2x-9)^2}$$
$$\equiv \frac{-56}{(2x-9)^2}.$$

This is non-zero, so there are no SPs.

(c) The graph is a transformed reciprocal $y = \frac{1}{x}$:



4234. Rewriting over base 2,

$$2^{\sin x} = 4^{\cos x}$$
$$\Longrightarrow 2^{\sin x} = 2^{2\cos x}$$
$$\Longrightarrow \sin x = 2\cos x$$
$$\Longrightarrow \tan x = 2.$$

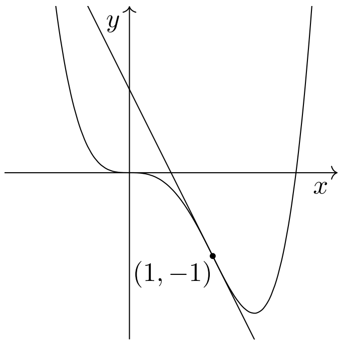
So, for $x \in [0, \pi)$, $x = \arctan 2$.

4235. (a) Solving for intersections,

$$\begin{aligned} & x^4 - 2x^3 = -2x + 1 \\ \Longrightarrow\ & x^4 - 2x^3 + 2x - 1 = 0 \\ \Longrightarrow\ & (x+1)(x-1)^3 = 0. \end{aligned}$$

Since the root at $x = 1$ is repeated, $(1, -1)$ is a point of tangency.

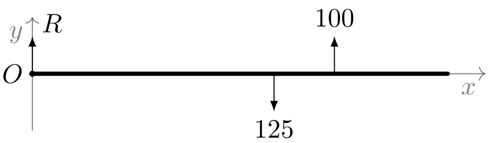
(b) The root at $x = 1$ is a triple root. Since the parity of the root is odd (multiplicity 3), curve and line cross at their point of tangency.



4236. Using a compound-angle formula,

$$\begin{aligned} \tan \tfrac{5\pi}{12} &= \tan\left(\tfrac{\pi}{6} + \tfrac{\pi}{4}\right) \\ &= \frac{\tan \frac{\pi}{6} + \tan \frac{\pi}{4}}{1 - \tan \frac{\pi}{6} \tan \frac{\pi}{4}} \\ &= \frac{\frac{\sqrt{3}}{3} + 1}{1 - \frac{\sqrt{3}}{3}} \\ &= \frac{\left(\frac{\sqrt{3}}{3} + 1\right)^2}{1 - \frac{1}{3}} \\ &= 2 + \sqrt{3}, \text{ as required.} \end{aligned}$$

4237. (a) The (x, y) plane is (horizontal, horizontal). The vertical z axis is represented by the origin O. Since there are no other forces acting in the x direction, the reaction force at the hinge acts only in the y direction. In plan view, the forces on the door are:



(b) Hinges ensure that a door can only rotate, not translate. So, we need only consider moments about the z axis. The forces are exerted at distances 80 cm and 64 cm from the hinges. Hence, the total moment around the hinges (z axis) is $100 \times 80 - 125 \times 64$. This is equal to zero, so the door remains in equilibrium.

(c) i. Moments around the point of contact of the 125 N force give $64R = 16 \times 100$. So, the component of $\mathbf{R}$ in the (x, y) plane is 25 N in the positive y direction.

ii. There is also a component of $\mathbf{R}$ in the z direction, which does not feature on the above diagram. It counteracts the weight, and has magnitude 200 N. In Newtons, the total contact force is

$$\mathbf{R} = \begin{pmatrix} 0 \\ 25 \\ 200 \end{pmatrix}.$$

4238. We can assume, without loss of generality, that the origin is the centre of rotational symmetry. Put into algebra, $\mathrm{h}(-x) = -\mathrm{h}(x)$. By the chain rule,

$$\begin{aligned} & \mathrm{h}(-x) = -\mathrm{h}(x) \\ \Longrightarrow\ & -\mathrm{h}'(-x) = -\mathrm{h}'(x) \\ \Longrightarrow\ & \mathrm{h}''(-x) = -\mathrm{h}''(x). \end{aligned}$$

The last line is the statement that the graph $y = \mathrm{h}''(x)$ has rotational symmetry around the origin. QED.

4239. (a) By the binomial expansion,

$$\begin{aligned} \left(1 + \frac{h}{x}\right)^{\frac{1}{3}} &= 1 + \tfrac{1}{3}\left(\frac{h}{x}\right) + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}\left(\frac{h}{x}\right)^2 + ... \\ &\equiv 1 + \frac{h}{3x} - \frac{h^2}{9x^2} + ... \end{aligned}$$

(b) Taking out a factor of $x^{\frac{1}{3}}$,

$$\begin{aligned} \frac{dy}{dx} &= \lim_{h \to 0} \frac{(x+h)^{\frac{1}{3}} - x^{\frac{1}{3}}}{h} \\ &\equiv \lim_{h \to 0} \frac{x^{\frac{1}{3}}\left(1 + \frac{x}{h}\right)^{\frac{1}{3}} - x^{\frac{1}{3}}}{h}. \end{aligned}$$

Substituting the result of part (a), this is

$$\begin{aligned} & \lim_{h \to 0} \frac{x^{\frac{1}{3}}\left(1 + \frac{h}{3x} - \frac{h^2}{9x^2} + ...\right) - x^{\frac{1}{3}}}{h} \\ \equiv\ & \lim_{h \to 0} \frac{\frac{h}{3}x^{-\frac{2}{3}} - \frac{h^2}{9}x^{-\frac{5}{3}} + ...}{h} \\ \equiv\ & \lim_{h \to 0} \tfrac{1}{3}x^{-\frac{2}{3}} - \tfrac{h}{9}x^{-\frac{5}{3}} + ... \end{aligned}$$

Taking the limit, the second and all subsequent terms tend to zero, leaving $\frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}}$. □

4240. Let $y = a$ and $b = x$.

Consider the boundary equation of the inequality, $y = x^2 - 4$. The equation for intersections of this with $y = x^4$ is

$$x^4 - x^2 + 4 = 0.$$

This is a quadratic in x^2. It has $\Delta = -13 < 0$, so has no real roots. Hence, the quartic curve $y = x^4$ does not intersect the parabola $y = x^2 - 4$, and must lie above it. So, if (x, y) is a point satisfying $y = x^4$, then (x, y) must also satisfy $y > x^2 - 4$. Written in the original algebra, this is

$$a = b^4 \implies a > b^2 - 4, \text{ as required.}$$

4241. If the counters are collinear, the line on which they lie is horizontal, vertical, or has gradient ± 1.

① HORIZONTAL. There are four lines to choose from. For each line, there are ${}^4\text{C}_3 = 4$ ways of choosing the locations of the counters. This gives 16 possibilities.

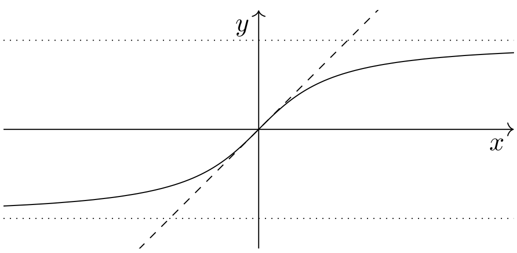
② VERTICAL. As above, 16 possibilities.

③ GRADIENT 1. The main diagonal (length 4) offers ${}^4\text{C}_3 = 4$ possibilities. The adjacent diagonals (length 3) each offer 1 possibility. This gives 6 overall.

④ GRADIENT -1. As above, 6 possibilities.

The total is $16 + 16 + 6 + 6 = 44$, as required.

4242. The graph of $y = \arctan x$, with its tangent at the origin and its two horizontal asymptotes, is as follows:



The boundary cases for $y = kx$ are the tangent shown, which has gradient 1, and the x axis, which has gradient 0. For $0 < k < 1$, $y = kx$ intersects the curve three times. For all other values of k, the origin is the only point of intersection. This gives $k \in (-\infty, 0] \cup [1, \infty)$.

4243. Rearranging the inequality,

$$\begin{aligned} &\mathbb{P}(X_1 < 1 - X_2) \\ \equiv\ &\mathbb{P}(X_1 + X_2 < 1). \end{aligned}$$

Since X_1 and X_2 are independent, the variable $Y = X_1 + X_2$ is normally distributed, with mean 0 and variance 2. So, with $Y \sim N(0, 2)$, using the normal distribution facility on a calculator,

$$\mathbb{P}(Y < 1) = 0.760 \text{ (3sf)}.$$

4244. The quantity $x^2 + y^2$ is squared distance from the origin of the (x, y) plane. So, we need the shortest distance between the origin and the line $ax + by = c$. This is along the normal $bx - ay = 0$. Solving simultaneously,

$$\begin{aligned} & ax + b \cdot \tfrac{b}{a} x = c \\ \Longrightarrow\ & a^2 x + b^2 x = ac \\ \Longrightarrow\ & x = \frac{ac}{a^2 + b^2}. \end{aligned}$$

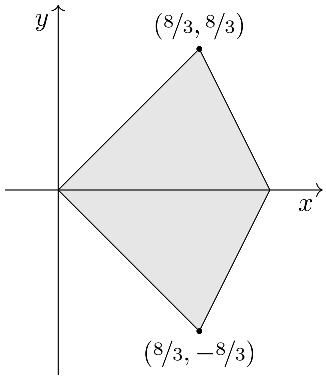
Substituting this back in,

$$y = \frac{bc}{a^2 + b^2}.$$

At this point, the value of $x^2 + y^2$ is

$$\begin{aligned} & \frac{a^2 c^2}{(a^2 + b^2)^2} + \frac{b^2 c^2}{(a^2 + b^2)^2} \\ \equiv\ & \frac{(a^2 + b^2) c^2}{(a^2 + b^2)^2} \\ \equiv\ & \frac{c^2}{a^2 + b^2}, \text{ as required.} \end{aligned}$$

4245. The result follows from the symmetry of the graphs. Both $x = |y|$ and $x = 4 - \frac{1}{2}|y|$ have the x axis as a line of symmetry, and each consists of two straight line segments. They intersect at $(8/3, 8/3)$, forming a quadrilateral with a line of symmetry through two of the vertices.



Such a shape is necessarily a kite.

4246. Breaking the position vector apart,

$$\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 4 \end{pmatrix} + t \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t^2 \begin{pmatrix} 0 \\ -4 \\ 0 \\ 3 \end{pmatrix}.$$

Ignoring the initial position, which isn't relevant, the displacement from the initial position is

$$\mathbf{s} = t \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t^2 \begin{pmatrix} 0 \\ -4 \\ 0 \\ 3 \end{pmatrix}.$$

Converting to unit vectors for subsequent clarity,

$$\mathbf{s} = \sqrt{5} t \begin{pmatrix} \frac{2}{\sqrt{5}} \\ 0 \\ \frac{1}{\sqrt{5}} \\ 0 \end{pmatrix} + 5t^2 \begin{pmatrix} 0 \\ -\frac{4}{5} \\ 0 \\ \frac{3}{5} \end{pmatrix}.$$

The unit vectors are perpendicular, because they have no components in common. Naming them $\mathbf{i}'$ and $\mathbf{j}'$, the path is

$$\mathbf{s} = \sqrt{5} t \mathbf{i}' + 5t^2 \mathbf{j}'.$$

This is a parabolic path in the plane containing the perpendicular unit vectors $\mathbf{i}'$ and $\mathbf{j}'$.

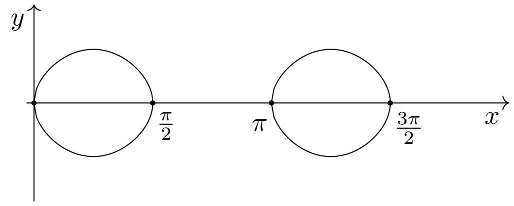
4247. Using a double-angle formula

$$y^2 = \tfrac{1}{2}\sin 2x.$$

The LHS is always positive, so wherever $\sin 2x < 0$, there are no points. Where $\sin 2x \geq 0$, we can take the square root:

$$y = \pm\sqrt{\tfrac{1}{2}\sin 2x}.$$

At roots of $\sin 2x$, the tangent is parallel to the y axis. The maxima are at $y = 1$. So, the graph of $y^2 = \sin x \cos x$ is



Despite appearances, these are not circles.

4248. Taking the positive square root,

$$\begin{aligned} y &= \tfrac{1}{2}\sqrt{x^2+9} \\ \implies \frac{dy}{dx} &= \frac{x}{2\sqrt{x^2+9}}. \end{aligned}$$

At $x = 4$, the gradient is $\frac{2}{5}$. This gives angle of projection $\theta = \arctan\frac{2}{5}$. The components of initial velocity are

$$\begin{aligned} u_x &= 5\sqrt{29}\cos\theta = 25, \\ u_y &= 5\sqrt{29}\sin\theta = 10. \end{aligned}$$

At $t = 0$, the skier is at $\left(4, \frac{5}{2}\right)$. So,

$$\begin{aligned} x &= 4 + 25t, \\ y &= \tfrac{5}{2} + 10t - 5t^2. \end{aligned}$$

Eliminating t,

$$\begin{aligned} y &= \tfrac{5}{2} + 10\left(\frac{x-4}{25}\right) - 5\left(\frac{x-4}{25}\right)^2 \\ \implies 250y &= 193 + 116x - 2x^2, \text{ as required.} \end{aligned}$$

4249. Using the factorial formula,

$$\begin{aligned} &{}^n\mathrm{C}_r + {}^n\mathrm{C}_{r+1} \\ \equiv\ & \frac{n!}{r!(n-r)!} + \frac{n!}{(r+1)!(n-r-1)!} \\ \equiv\ & \frac{n!(r+1)}{(r+1)!(n-r)!} + \frac{n!(n-r)}{(r+1)!(n-r)!} \\ \equiv\ & \frac{n!(n+1)}{(r+1)!(n-r)!} \\ \equiv\ & \frac{(n+1)!}{(r+1)!(n+1-(r+1))!} \\ \equiv\ & {}^{n+1}\mathrm{C}_{r+1}. \end{aligned}$$

This is the entry between ${}^n\mathrm{C}_r$ and ${}^n\mathrm{C}_{r+1}$, in the row below them. Hence, the formula given satisfies the addition property of Pascal's triangle. □

4250. Using the identity $\tan\theta \equiv \cot(90° - \theta)$, we can express $\tan 50°$ as $\cot 40°$. This gives

$$\tan 10° \times \cot 40° = \tan 20° \times \tan 30°.$$

Multiplying both sides by $\tan 40°$ gives

$$\tan 10° = \tan 20° \times \tan 30° \times \tan 40°, \text{ as required.}$$

4251. (a) Quoting $\sin\theta \approx \theta$, $a = 1$.

(b) Setting up equality and differentiating twice,

$$\begin{aligned} \sin\theta &= \theta + b\theta^2 + c\theta^3 \\ \implies \cos\theta &= 1 + 2b\theta + 3c\theta^2 \\ \implies -\sin\theta &= 2b + 6c\theta. \end{aligned}$$

Evaluating at $\theta = 0$, $b = 0$.

(c) Equating the third derivatives, $-\cos\theta = 6c$. At $\theta = 0$, this is $-1 = 6c$, so $c = -1/6$. Hence, the cubic approximation is

$$\sin\theta \approx \theta - \tfrac{1}{6}\theta^3.$$

4252. Putting each pair over a common denominator,

$$\begin{aligned} &\frac{1}{1+2x} + \frac{1}{1-2x} + \frac{1}{2+x} + \frac{1}{2-x} = 0 \\ \implies & \frac{2}{1-4x^2} + \frac{4}{4-x^2} = 0 \\ \implies & 2\left(4-x^2\right) + 4\left(1-4x^2\right) = 0 \\ \implies & x = \pm\sqrt{\tfrac{2}{3}}. \end{aligned}$$

4253. (a) For $k \in (0, 1)$, the function $x \mapsto x^{-k}$ grows asymptotically as $x \to 0$. So, the integral must be calculated as a limit:

$$\begin{aligned} \mathrm{A}(k) &= \lim_{p\to 0^+} \int_p^1 x^{-k}\,dx \\ &\equiv \lim_{p\to 0^+} \Big[\tfrac{1}{1-k}x^{1-k}\Big]_p^1 \\ &\equiv \lim_{p\to 0^+} \left(\tfrac{1}{1-k} - \tfrac{1}{1-k}p^{1-k}\right). \end{aligned}$$

Since $1 - k > 0$, the right-hand term tends safely to zero as we take the limit. Hence, for $k \in (0, 1)$, the function A is well defined:

$$\mathrm{A}(k) = \tfrac{1}{1-k}.$$

(b) For $k = 1$, the function $x \mapsto x^{-k}$ also grows asymptotically as $x \to 0$. So, the integral must again be calculated as a limit:

$$\begin{aligned} \mathrm{A}(k) &= \lim_{p\to 0^+} \int_p^1 x^{-1}\,dx \\ &\equiv \lim_{p\to 0^+} \Big[\ln|x|\Big]_p^1 \\ &\equiv \lim_{p\to 0^+} -\ln p. \end{aligned}$$

This limit diverges to positive infinity, so the function $\mathrm{A}(k)$ is not well defined at $k = 1$.

4254. By the factor theorem, $g(x)$ has a factor of $(x-\alpha)$. So, $g(x) = (x-\alpha)\,p(x)$, where p is a polynomial function. By the product rule,

$$g'(x) = p(x) + (x-\alpha)\,p'(x).$$

Substituting $x = \alpha$ gives $g'(\alpha) = p(\alpha)$, which tells us that $p(\alpha) = 0$. Hence, $p(x)$ must have a factor of $(x-\alpha)$. We can now write $g'(x) = (x-\alpha)\,q(x)$. Differentiating again,

$$g''(x) = q(x) + (x-\alpha)\,q'(x).$$

Evaluating at $x = \alpha$ gives $q(\alpha) = 0$, so $q(x)$ has a factor of $(x-\alpha)$. This implies that $g'(x)$ has a factor of $(x-\alpha)^2$ and that $g(x)$ has a factor of $(x-\alpha)^3$. □

4255. By the first Pythagorean identity, the value is 1.

4256. At point (p, p^2), the normal has equation

$$y - p^2 = -\tfrac{1}{2p}(x-p).$$

Solving this simultaneously with $y = x^2$,

$$\begin{aligned} x^2 - p^2 &= -\tfrac{1}{2p}(x-p) \\ \Longrightarrow 2px^2 + x - 2p^3 - p &= 0 \\ \Longrightarrow x &= \frac{-1 \pm \sqrt{1 + 8p(2p^3+p)}}{4p} \\ &\equiv \frac{-1 \pm \sqrt{(4p^2+1)^2}}{4p} \\ &\equiv p, -\tfrac{1}{2p} - p. \end{aligned}$$

The latter is the x coordinate of P. For the least value of this expression (x coordinate closest to O, thus least value of y), we set its derivative to zero:

$$\begin{aligned} \frac{1}{2p^2} - 1 &= 0 \\ \Longrightarrow p &= \pm\tfrac{\sqrt{2}}{2}. \end{aligned}$$

At $p = \frac{\sqrt{2}}{2}$, the x coordinate of P is $-\sqrt{2}$, giving $y = 2$. Hence, the minimum possible value of the y coordinate of P is 2. □

4257. The compound-angle formula is

$$\tan(\theta + \phi) \equiv \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}.$$

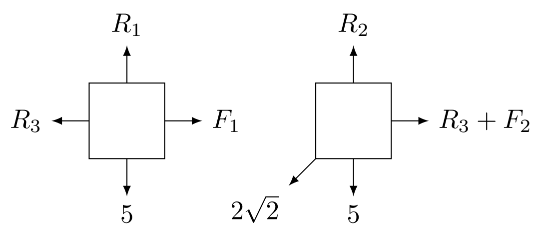
Setting $\theta = \theta_1$ and $\phi = \theta_2 + \theta_3$, and using the notation given in the question,

$$\begin{aligned} &\tan(\theta_1 + \theta_2 + \theta_3) \\ \equiv\ &\frac{x + \tan(\theta_2+\theta_3)}{1 - x\tan(\theta_2+\theta_3)}. \end{aligned}$$

Using the compound-angle formula again, this is

$$\begin{aligned} &\frac{x + \frac{y+z}{1-yz}}{1 - x\frac{y+z}{1-yz}} \\ \equiv\ &\frac{x(1-yz) + y + z}{1 - yz - x(y+z)} \\ \equiv\ &\frac{x+y+z-xyz}{1-xy-yz-xz}, \text{ as required.} \end{aligned}$$

4258. (a) The force diagrams are



Resolving vertically, $R_1 = 5$ and $R_2 = 7$. So, maximal friction is 5μ for the left-hand block and 7μ for the right-hand block. For the whole system, horizontal equilibrium gives $F_1 + F_2 = 2$. Consider two cases:

① If friction is limiting for both blocks, then horizontal equilibrium gives $12\mu = 2$, so $\mu = \frac{1}{6}$.

② If friction is limiting for one block but not for the other, then the total friction is $F_1 + F_2 < 12\mu$. So $2 < 12\mu$, giving $\mu > \frac{1}{6}$.

So, the minimum possible value of μ is $\frac{1}{6}$.

(b) We are only told that the system of *the two blocks together* is in limiting equilibrium, which does not distinguish between the cases above. Examples are:

- $\mu = \frac{1}{6}$. Maximal frictions are $\frac{5}{6}$ N and $\frac{7}{6}$ N. These are sufficient to counteract the horizontal driving force of 2 N. The force R_3 between the blocks is $\frac{5}{6}$ N.
- $\mu = \frac{1}{5}$. Maximal frictions are 1 N and $\frac{7}{5}$ N. So, the left-hand block is in limiting equilibrium, and the force R_3 between the blocks is 1 N. This allows the frictional force F_2 on the right-hand block to be 1 N, which is less than maximal.

4259. (a) Solving for x intercepts,

$$\begin{aligned} \tan x - 2\sec x &= 0 \\ \Longrightarrow \sin x - 2 &= 0. \end{aligned}$$

This has no roots, as 2 is outside the range of the sine function. So, the curve does not cross the x axis.

(b) Setting the derivative to zero for SPs:

$$\begin{aligned} & \sec^2 x - 2\sec x \tan x = 0 \\ \Longrightarrow\ & \sec x(\sec x - 2\tan x) = 0 \\ \Longrightarrow\ & \sec x = 2\tan x \\ \Longrightarrow\ & \tfrac{1}{2} = \sin x. \end{aligned}$$

This gives SPs at $\left(\frac{\pi}{6}, -\sqrt{3}\right)$ and $\left(\frac{5\pi}{6}, \sqrt{3}\right)$. The second derivative is

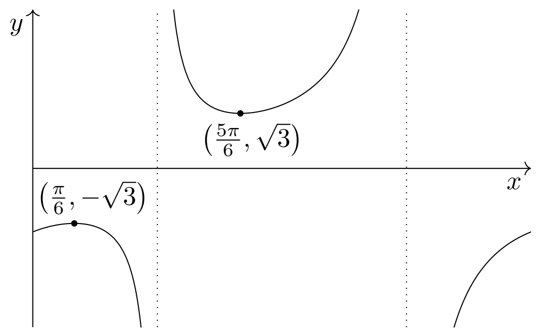
$$\frac{d^2y}{dx^2} = 2\sec^2 x \tan x - 2\sec x \tan^2 x - 2\sec^3 x.$$

Evaluating at the SPs,

x	$\frac{d^2y}{dx^2}$	SP
$\frac{\pi}{6}$	$-\frac{4}{\sqrt{3}}$	Max
$\frac{5\pi}{6}$	$\frac{4}{\sqrt{3}}$	Min

(c) The curve has vertical asymptotes where $\cos x = 0$, which is $x = \pi/2, 3\pi/2$.

(d) The y intercept is -2. Joining the dots, the curve is



4260. Call the cubic $y = \mathrm{f}(x)$. We know that $\mathrm{f}'(x)$ is a quadratic. Since there are stationary points at $x = p, q$, this quadratic has roots at $x = p, q$. Hence, $\mathrm{f}'(x) = k(ax^2 + bx + c)$, for some constant k. Integrating this,

$$\mathrm{f}(x) = \tfrac{1}{3}akx^3 + \tfrac{1}{2}bkx^2 + ckx + d.$$

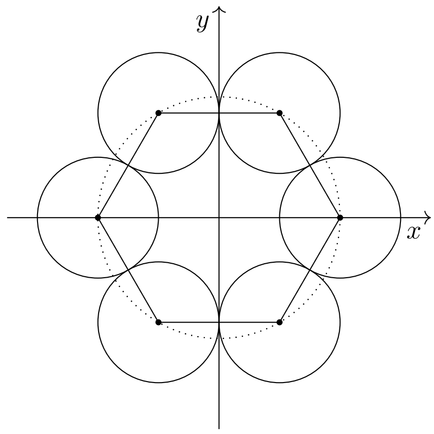
The curve $y = \mathrm{f}(x)$ passes through the origin, so $d = 0$. And it is monic, so $\frac{1}{3}ak = 1$. This gives $k = \frac{3}{a}$. Hence, the equation of the cubic is

$$y = x^3 + \tfrac{3b}{2a}x^2 + \tfrac{3c}{a}x.$$

4261. The centres of the circles are at

$$\left(2\cos\frac{k\pi}{3}, 2\sin\frac{k\pi}{3}\right).$$

The position vectors of the centres have length 2, and their directions (anticlockwise from positive x) are $\theta = 0, \frac{\pi}{3}, ...$ This puts the centres at the vertices of a regular hexagon of side length 2. The circles have radius 1, which means that each pair of adjacent circles is tangent at the midpoint of the edges of the following hexagon:



4262. (a) If the quartic has a triple root at $x = \alpha$, then it has factor of $(x - \alpha)^3$. Taking this cubic out of a quartic must leave a single factor, which corresponds to a single root.

(b) By the binomial expansion,

$$(x + b)^3 \equiv x^3 + 3x^2b + 3xb^2 + b^3.$$

So, the full expansion is

$$\begin{aligned} & ax^4 + (3ab + ac)x^3 + (3ab^2 + 3abc)x^2 \\ & \qquad + (ab^3 + 3ab^2c)x + ab^3c. \end{aligned}$$

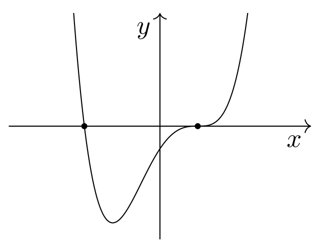
(c) Equating coefficients of x^4, $a = 8$. Equating coefficients of x^3 and x^2,

$$\begin{aligned} 24b + 8c = -4 &\Longrightarrow 6b + 2c = -1, \\ 24b^2 + 24bc = -6 &\Longrightarrow 4b^2 + 4bc = -1. \end{aligned}$$

Substituting for c,

$$\begin{aligned} & 4b^2 + 4b\left(\tfrac{-1-6b}{2}\right) = -1 \\ \Longrightarrow\ & b = -\tfrac{1}{2}, \tfrac{1}{4}. \end{aligned}$$

These give $c = 1$ and $c = -\frac{5}{4}$ respectively. Checking the coefficient of x^0, the second set of values don't work. So, $b = -\frac{1}{2}$ and $c = 1$. The quartic has a triple root at $x = \frac{1}{2}$ and a single root at $x = -1$.



4263. We know that

$$\int_0^k \mathrm{f}(x)\,dx + \int_{-k}^0 \mathrm{f}(x)\,dx = 0.$$

We can reverse the direction of the second integral, which corresponds to evaluating the same area in the opposite direction. This gives

$$\int_0^k \mathrm{f}(x)\,dx + \int_0^k \mathrm{f}(-x)\,dx = 0.$$

Combining the integrals,

$$\int_0^k \mathrm{f}(x) + \mathrm{f}(-x)\,dx = 0.$$

The integrand is a polynomial. The area beneath it between $x = 0$ and $x = k$ is zero for all k, so it must be the zero polynomial. In other words, $\mathrm{f}(x) + \mathrm{f}(-x) = 0$ for all x. Hence, $\mathrm{f}(-x) = -\mathrm{f}(x)$, meaning that $y = \mathrm{f}(x)$ has rotational symmetry around the origin. □

— Nota Bene —

The initial step can be shown more explicitly by using the substitution $X = -x$. The minus sign in $dX = -dx$ cancels with the minus sign which appears when reversing the order of the limits.

4264. Differentiating, $\mathrm{g}'(x) = ae^{ax} - be^{bx}$. Setting this to zero for SPs,

$$\begin{aligned} &ae^{ax} - be^{bx} = 0 \\ \implies &e^{ax}\left(a - be^{(b-a)x}\right) = 0. \end{aligned}$$

The first factor cannot be zero. So,

$$\begin{aligned} &a - be^{(b-a)x} = 0 \\ \implies &e^{(b-a)x} = \tfrac{a}{b} \\ \implies &(b-a)x = \ln\tfrac{a}{b} \\ \implies &x = \tfrac{1}{b-a}\ln\tfrac{a}{b}. \end{aligned}$$

If the above solution to $\mathrm{g}'(x) = 0$ exists, then g has a stationary value. There are two conditions to consider:

① The above solution requires $a \neq b$. However, in this case, the function g is identically zero, thus stationary, everywhere.

② The above solution requires $\frac{a}{b} > 0$, which is equivalent to saying that a and b must either both be positive or both be negative.

Summarising the above, $\mathrm{g}(x)$ has a stationary value iff a and b are both positive, both negative, or both zero.

4265. (a) Substituting the x and y components, the first Pythagorean trig identity gives

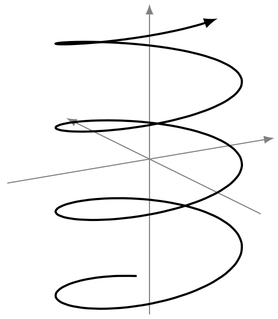
$$x^2 + y^2 = \cos^2 2t + \sin^2 2t = 1.$$

In the (x, y) plane, this is a unit circle. In a 3D space, with no restriction on z, this unit circle generates the curved surface of a cylinder.

(b) Differentiating,

$$\begin{aligned} &\mathbf{r} = \cos 2t\mathbf{i} + \sin 2t\mathbf{j} + 5t\mathbf{k} \\ \implies &\mathbf{v} = -2\sin 2t\mathbf{i} + 2\cos 2t\mathbf{j} + 5\mathbf{k} \\ \implies &|\mathbf{v}| = \sqrt{4\sin^2 2t + 4\cos^2 2t + 25} \\ \implies &|\mathbf{v}| = \sqrt{29}. \end{aligned}$$

(c) In the (x, y) plane, the motion is circular. This is combined with constant velocity in z, which gives a helix or corkscrew.



4266. (a) The image of $(1, 0)$ is

$$\left(\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}\right).$$

The distance from the origin is

$$\sqrt{\frac{a^2}{a^2+b^2} + \frac{b^2}{a^2+b^2}} \equiv \sqrt{\frac{a^2+b^2}{a^2+b^2}} \equiv 1.$$

(b) Multiplying by a and b,

$$aX = \frac{a^2x + aby}{\sqrt{a^2+b^2}}, \quad bY = \frac{b^2x - aby}{\sqrt{a^2+b^2}}.$$

Adding these, the terms in y cancel:

$$\begin{aligned} &aX + bY = \frac{(a^2+b^2)x}{\sqrt{a^2+b^2}} \\ \implies &x = \frac{aX + bY}{\sqrt{a^2+b^2}}. \end{aligned}$$

Repeating the procedure,

$$\begin{aligned} &bX - aY = \frac{(a^2+b^2)y}{\sqrt{a^2+b^2}} \\ \implies &y = \frac{bX - aY}{\sqrt{a^2+b^2}}. \end{aligned}$$

4267. The possibility space consists of 6^6 outcomes. There are ${}^6\mathrm{C}_3 = 20$ sets of three different scores. For each of these, there are $\frac{6!}{2!2!2!} = 90$ different orders. So, the probability is

$$p = \frac{20 \times 90}{6^6} = \tfrac{25}{648}.$$

— Alternative Method —

The probability that the scores are AABBCC, in that order, is

$$1 \times \tfrac{1}{6} \times \tfrac{5}{6} \times \tfrac{1}{6} \times \tfrac{4}{6} \times \tfrac{1}{6} = \tfrac{5}{1944}.$$

The number of orders of AABBCC is

$$\frac{6!}{2!2!2!} = 90.$$

So, the number of orders of AABBCC in which e.g. AABBCC and BBAACC are counted as one is

$$\frac{6!}{2!2!2! \times 3!} = 15.$$

This gives $p = 15 \times \frac{5}{1944} = \frac{25}{648}$.

4268. Label the squares as follows

1	2	3
4	5	6
7	8	9

Let × start, without loss of generality. Consider the case that ○ plays an even square. Without loss of generality, say ○ plays 2.

	○	
	×	

Then, if × plays 1, ○ is forced to play 9.

×	○	
	×	
		○

× then plays 7, leaving ○ with two lines of three to cover, in squares 3 and 4.

×	○	
	×	
×		○

○ cannot play in both shaded squares, so has lost. Hence, if the first player plays in the middle, the second player must play in a corner. □

4269. (a) Differentiating the sum,

$$\begin{aligned} \mathrm{E}(x) &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ... \\ \Longrightarrow \mathrm{E}'(x) &= \frac{1}{1!} + \frac{2x}{2!} + \frac{3x^2}{3!} + ... \\ &\equiv 1 + x + \frac{x^2}{2!} + ... \\ &= \mathrm{E}(x). \end{aligned}$$

(b) Let $y = \mathrm{E}(x)$.

$$\begin{aligned} &\frac{dy}{dx} = y \\ \Longrightarrow &\int \frac{1}{y}\,dx = \int 1\,dx \\ \Longrightarrow &\ln|y| = x + c \\ \therefore \quad &y = Ae^x. \end{aligned}$$

So, the general solution is $\mathrm{E}(x) = Ae^x$. Since $\mathrm{E}(1) = e$, $A = 1$, which means that E is the exponential function $x \mapsto e^x$.

(c) From part (b), we know that

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

Substituting $x = 1$,

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ...$$

Splitting the sum into two parts,

$$\begin{aligned} e &= \sum_{i=1}^{4} \frac{1}{i!} + \sum_{i=5}^{\infty} \frac{1}{i!} \\ \Longrightarrow \sum_{i=5}^{\infty} \frac{1}{i!} &= e - \sum_{i=1}^{4} \frac{1}{i!}. \end{aligned}$$

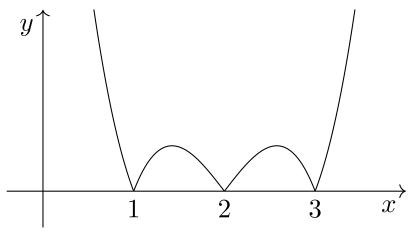
The latter sum is $\frac{65}{24}$. So,

$$\begin{aligned} \sum_{i=5}^{\infty} \frac{1}{i!} &= e - \frac{65}{24} \\ &= 0.009948... \\ &< \frac{1}{100}, \text{ as required.} \end{aligned}$$

4270. The RHS is positive. In the original equation, each factor is individually positive. But this is the same as making the whole product positive. So, consider

$$y = |(x-1)(x-2)(x-3)|.$$

Removing the mod signs, $y = (x-1)(x-2)(x-3)$ is a positive cubic with single roots at $x = 1, 2, 3$. We then apply a mod function, making all negative y values positive. Hence, the curve is



4271. Since the vertical asymptote is at $x = -1$, this is a root of the denominator. So, $c = 4$. As $x \to \infty$, the curve tends to the oblique asymptote $y = \frac{1}{2}x - \frac{1}{4}$. So, the equation of the curve must be expressible, for some constant k, as

$$y = \tfrac{1}{2}x - \tfrac{1}{4} + \frac{k}{4x+4}.$$

Putting this over a common denominator,

$$\begin{aligned} y &= \frac{\left(\frac{1}{2}x - \frac{1}{4}\right)(4x+4) + k}{4x+4} \\ &\equiv \frac{2x^2 + x - 1 + k}{4x+4}. \end{aligned}$$

So, $a = 2$ and $b = 1$ (and $k = 1$). The equation of the curve is

$$y = \frac{2x^2 + x}{4x+4}.$$

4272. Rearranging and squaring the first equation, we have $\sin^2 x = 2\cos^2 y$. The first Pythagorean trig identity gives

$$\begin{aligned} 1-\cos^2 x &= 2(1-\sin^2 y) \\ \Longrightarrow \cos^2 x &= 2\sin^2 y - 1. \end{aligned}$$

Rearranging and squaring the second equation,

$$\begin{aligned} 2\cos^2 x &= 16 - 16\sqrt{3}\sin y + 12\sin^2 y \\ \Longrightarrow \cos^2 x &= 8 - 8\sqrt{3}\sin y + 6\sin^2 y. \end{aligned}$$

Substituting for $\cos^2 x$,

$$\begin{aligned} & 2\sin^2 y - 1 = 8 - 8\sqrt{3}\sin y + 6\sin^2 y \\ \Longrightarrow\ & 4\sin^2 y - 8\sqrt{3}\sin y + 9 = 0 \\ \Longrightarrow\ & \sin y = \tfrac{\sqrt{3}}{2}, \tfrac{3\sqrt{3}}{2}. \end{aligned}$$

The latter has no roots, as $3\sqrt{3}/2 > 1$. The former gives $y = \pi/3, 2\pi/3$. Substituting back in, the (x, y) pairs are $(\pi/4, \pi/3)$ and $(7\pi/4, 2\pi/3)$.

4273. We are given $u_1 = a$ and $u_n = b$. If $a = b = 0$, then the sequence is the zero sequence and all terms are trivially known. So, we can assume that $a, b \neq 0$.

Using the ordinal formula for a GP, $b = ar^{n-1}$, which we can rearrange to $r^{n-1} = \frac{b}{a}$. Call this equation E. We know the values of a, b and n, but cannot necessarily solve E for r. Consider the parity of n:

- If n is even, then $n-1$ is odd and so E has exactly one real root. Any term of the GP can therefore be calculated with certainty.
- If n is odd, then $n-1$ is even and E has exactly two real roots $r = \pm R$. Nevertheless, r^2 is known with certainty. So, since we know the first term, we also know the third term, and every subsequent odd-numbered term.

In both cases, we know terms of the form u_{2k+1}, for $k \in \mathbb{N}$, with certainty. □

4274. The magnitude of the vector is

$$\begin{aligned} &\sqrt{\tan^4 t + 3\sec^2 t} \\ \equiv\ &\sqrt{\tan^4 t + 3(1+\tan^2 t)} \\ \equiv\ &\sqrt{\tan^4 t + 3\tan^2 t + 3}. \end{aligned}$$

Since $\tan^4 t$ and $\tan^2 t$ are both squares, each is non-negative. So, the radicand is never less than 3. Hence, $|\mathbf{b}|$ is never less than $\sqrt{3}$.

— Alternative Method —

The range of the sec function is $(-\infty, -1] \cup [1, \infty)$. So, $|\sec t| \geq 1$ for all t. Hence, the $\mathbf{j}$ component of $\mathbf{b}$ has magnitude at least $\sqrt{3}$.

Adding an $\mathbf{i}$ component to this can only increase the magnitude. So, $|\mathbf{b}| \geq \sqrt{3}$, as required.

4275. Classifying by the largest number of black beads n in a single group, the arrangements, with the white counters represented as dots, are as follows:

n	4	3	2	1
	BBBB....	BBB.B...	BB.BB...	B.B.B.B.
		BBB..B..	BB..BB..	
			BB.B.B..	
			BB.B..B.	

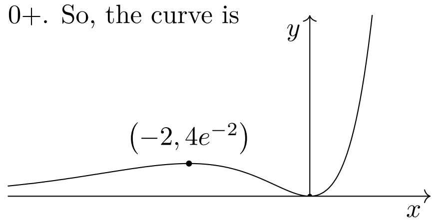
All other arrangements are reflections or rotations of one of the above. So, there are eight bracelets.

4276. (a) Using the product rule,

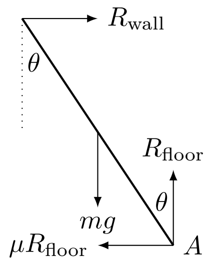
$$\begin{aligned} & y = x^2 e^x \\ \Longrightarrow\ & \frac{dy}{dx} = \big(2x + x^2\big)e^x \\ \Longrightarrow\ & \frac{d^2y}{dx^2} = \big(2 + 4x + x^2\big)e^x. \end{aligned}$$

Setting the first derivative to zero, there are SPs at $x = 0, -2$. The second derivatives are respectively 2 and $-2e^{-2}$. So, there is a local minimum at the origin, and a local maximum at $\big(-2, 4e^{-2}\big)$.

(b) The only axis intercept is a double root at the origin. As $x \to \infty$, $y \to \infty$. And as $x \to -\infty$, $y \to 0+$. So, the curve is



4277. (a) With the ladder at its greatest angle to the vertical, friction is limiting: $F_{\text{max}} = \mu R_{\text{floor}}$. The force diagram is



Let the ladder have length 2 m.

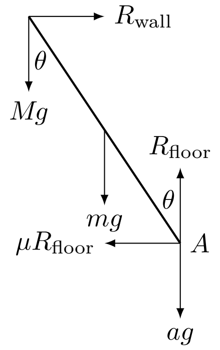
$$\begin{aligned} \updownarrow &: R_{\text{floor}} - mg = 0 \\ \leftrightarrow &: R_{\text{wall}} - \mu R_{\text{floor}} = 0 \\ \overset{\frown}{A} &: R_{\text{wall}} \cdot 2\cos\theta - mg \cdot 1\sin\theta = 0. \end{aligned}$$

The first two equations give $R_{\text{floor}} = mg$ and $R_{\text{wall}} = \mu mg$. Rearranging the third,

$$\tan\theta = \frac{2R_{\text{wall}}}{mg} = \frac{2\mu mg}{mg} = 2\mu.$$

So, the greatest angle is $\arctan 2\mu$.

(b) Let the ladder be in limiting equilibrium, with a person of mass M at the top, a load of mass a at A, and θ as in part (a).



The equations are

$$\begin{aligned}
\updownarrow &: R_{\text{floor}} - mg - Mg - ag = 0 \\
\leftrightarrow &: R_{\text{wall}} - \mu R_{\text{floor}} = 0 \\
\overset{\curvearrowright}{A} &: 2R_{\text{wall}}\cos\theta - 2Mg\sin\theta - mg\sin\theta = 0.
\end{aligned}$$

Rearranging the moments equation,

$$\begin{aligned}
2R_{\text{wall}}\cos\theta &= \sin\theta(2Mg + mg) \\
\implies R_{\text{wall}} &= \frac{2Mg + mg}{2}\tan\theta \\
&= (2M + m)\mu g.
\end{aligned}$$

So, $R_{\text{floor}} = (2M + m)g$. Substituting this into the vertical equation,

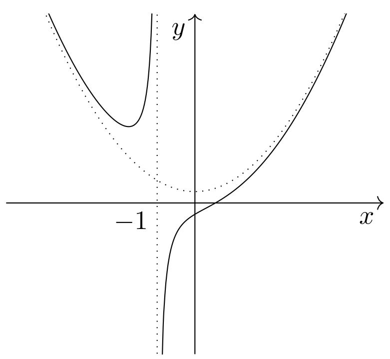
$$\begin{aligned}
&(2M + m)g - mg - Mg - ag = 0 \\
\implies &a = M.
\end{aligned}$$

So, to allow a person of mass M to reach the top, a load of mass M must be fixed at the bottom.

4278. Rewriting the second term,

$$\frac{x-1}{x+1} \equiv \frac{x+1-2}{x+1} \equiv 1 - \frac{2}{x+1}.$$

This tends to 1 as $x \to \pm\infty$. The curve approaches $y = x^2 + 1$ from below as $x \to \infty$ and from above as $x \to \infty$. In between, it has a single asymptote at $x = -1$. So, the curve is



4279. The faces are equilateral triangles, with three edges. So, the ant can't return to A by walking the perimeter of a face. The only paths that get the ant back to A circumnavigate the octahedron. They are "*great squares*". (I refer here to the fact that e.g. the equator is a *great circle* of the globe.)

There are four such great square paths from A, one starting with each of B, D, E, F. The ant will certainly choose one of these vertices to start with. Suppose it is B, without loss of generality.

At B, the ant has three edges to choose from, of which one, C, is successful. At C, the ant again has three edges to choose from, of which one, D, is successful. At D, the ant again has three edges to choose from, of which one, A, is successful. So, the probability is $(1/3)^3 = 1/27$.

4280. Rearranging, the differential equation is

$$\frac{dx}{dt} = \frac{2x}{t} - t^2x^2.$$

By the quotient rule,

$$\begin{aligned}
x &= \frac{5t^2}{t^5 + c} \\
\implies \frac{dx}{dt} &= \frac{10t(t^5 + c) - 5t^2\big(5t^4\big)}{(t^2 + c)^2} \\
&\equiv \frac{10ct - 15t^6}{(t^2 + c)^2}.
\end{aligned}$$

The RHS of the differential equation is

$$\begin{aligned}
&\frac{2x}{t} - t^2x^2 \\
&= \frac{10t}{t^5 + c} - t^2\left(\frac{5t^2}{t^5 + c}\right)^2 \\
&\equiv \frac{10t(t^5 + c) - 25t^6}{(t^5 + c)^2} \\
&\equiv \frac{10ct - 15t^6}{(t^5 + c)^2}.
\end{aligned}$$

Hence, the proposed equation is a solution curve of the differential equation, as required.

4281. (a) Firstly, consider p^2 and q^2. These are

$$\begin{aligned}
p^2 &= x^2 + 2xy + y^2, \\
q^2 &= x^2 - 2xy + y^2.
\end{aligned}$$

Adding and subtracting these equations,

$$\begin{aligned}
p^2 + q^2 &= 2\big(x^2 + y^2\big), \\
p^2 - q^2 &= 4xy.
\end{aligned}$$

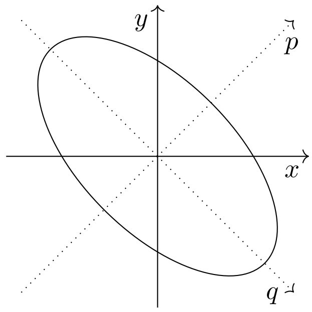
This gives

$$\begin{aligned}
x^2 + xy + y^2 &= \tfrac{1}{2}\big(p^2 + q^2\big) + \tfrac{1}{4}\big(p^2 - q^2\big) \\
&\equiv \tfrac{3}{4}p^2 + \tfrac{1}{4}q^2.
\end{aligned}$$

(b) The equation of the curve is

$$\tfrac{3}{4}p^2 + \tfrac{1}{4}q^2 = 1.$$

As a stretched version of $p^2+q^2 = 1$, this is the equation of an ellipse in the (p, q) plane. The variables p and q are proportional to position along the axes $y = \pm x$. So, the equation is an ellipse, whose major and minor axes are those lines:



4282. Call the upwards acceleration a. After time t, the height is $h = \frac{1}{2}at^2$, and the (upwards) velocity is $v = at$. Upon letting go at time t, these are the initial conditions for projectile motion. Landing speed V is given by

$$\begin{aligned} V^2 &= v^2 + 2gh \\ &= a^2t^2 + agt^2. \end{aligned}$$

Taking the positive square root,

$$V = t\sqrt{a^2 + ag} \propto t, \text{ as required.}$$

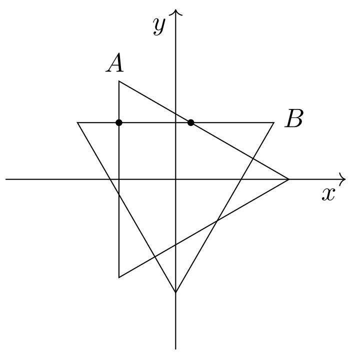
4283. In each case, the number of x intercepts and the number of vertical asymptotes correspond to the number of roots of the numerator and the number of roots of the denominator.

(a) The numerator has one root; the denominator has two roots. The graph has one x intercept and two vertical asymptotes.

(b) Both numerator and denominator have two roots. So, the graph has two x intercepts and two vertical asymptotes.

(c) The numerator has one root; the denominator has no (real) roots. So, the graph has one x intercept and no vertical asymptotes.

4284. The scenario, rotating 30° anticlockwise, is



(a) Consider the marked point of intersection in the second quadrant. The left-hand edge of the first triangle is parallel to the y axis. The top edge of the second triangle is parallel to the x axis. These edges are perpendicular. By symmetry, there are also two other matching points of intersection where the triangles meet at right-angles.

(b) Consider the other marked intersection. The direction of this point from O is the mean of the directions of A and B. Anticlockwise from the positive x axis, these are 120° and 30°, so the direction of the intersection is 75°. The y coordinate is the same as that of B, which is $\frac{1}{2}$. So, the distance from the origin is $\frac{1}{2}\operatorname{cosec} 75°$.

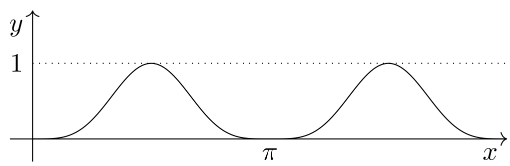
4285. Consider the function $f(x) = x^3 - x^2 + x$. Its derivative is

$$f'(x) = 3x^2 - 2x + 1.$$

This is a positive quadratic with $\Delta = -8 < 0$. So, it is always positive. Hence, the function f is increasing everywhere. Therefore, if $p > q$, then $f(p) > f(q)$. Written out in full, this is

$$p > q \implies p^3 - p^2 + p > q^3 - q^2 + q.$$

4286. The graph is akin to $y = \sin^2 x$, which, according to a double-angle formula, is $y = \frac{1}{2}(1 - \cos 2x)$. The difference is that the x intercepts of $y = \sin^4 x$ represent quadruple roots, as opposed to double roots. So, the minima are more snub-nosed than the maxima:



4287. (a) This cannot be determined with the info in the question. The fan-belt is a closed loop, so it could be tightened to arbitrary tension.

(b) As above, this cannot be determined.

(c) Taking NII around the fan-belt, the unknown tensions cancel, as do the weights of the two buckets. All that remains is the driving force and the weight of the liquid: $8g - mg = 0$. So, each full bucket carries 8 kg of liquid.

4288. This is a quadratic in $t^{\frac{1}{3}}$:

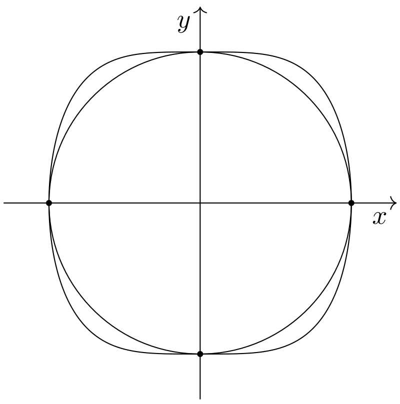
$$\begin{aligned} & t^{\frac{1}{3}} = 2 + 15t^{-\frac{1}{3}} \\ \implies & t^{\frac{2}{3}} - 2t^{\frac{1}{3}} - 15 = 0 \\ \implies & \left(t^{\frac{1}{3}} + 3\right)\left(t^{\frac{1}{3}} - 5\right) = 0 \\ \implies & t^{\frac{1}{3}} = -3, 5 \\ \implies & t = -27, 125. \end{aligned}$$

4289. Subtracting the two equations,

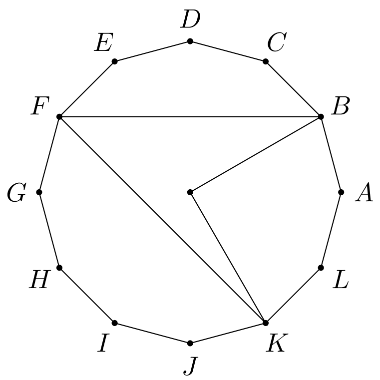
$$\begin{aligned} & x^4 - x^2 = 0 \\ \Longrightarrow\ & x^2(x+1)(x-1) = 0 \\ \Longrightarrow\ & x = 0, \pm 1. \end{aligned}$$

This gives four points of intersection at $(\pm 1, 0)$ and $(0, \pm 1)$.

To prove that these four are points of tangency, consider $x^4 + y^2 = 1$ in the positive quadrant. Since x^4 and y^2 are both positive, we know that $x \in [0, 1]$. Therefore, $x^4 \leq x^2$. Hence, any point on the curve $x^4 + y^2 = 1$ must lie on or outside the unit circle $x^2 + y^2 = 1$. This guarantees that the points of intersection are points of tangency.



4290. The scenario is



The angle at the centre is subtended by three out of twelve edges, and is therefore a right angle. So, by the angle at the centre theorem, $\angle BFK = 45°$.

4291. Since $x^2 \equiv |x|^2$, we can factorise as follows:

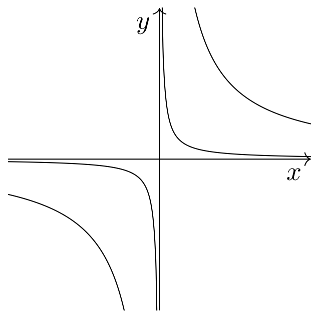
$$\begin{aligned} & x^2 + |x| - 6 = 0 \\ \Longrightarrow\ & |x|^2 + |x| - 6 = 0 \\ \Longrightarrow\ & \big(|x| + 3\big)\big(|x| - 2\big) = 0 \\ \Longrightarrow\ & |x| = -3, 2. \end{aligned}$$

The former gives no roots, so $x = \pm 2$.

4292. This is a quadratic in xy.

$$\begin{aligned} & xy + \frac{1}{xy} = 4 \\ \Longrightarrow\ & (xy)^2 - 4(xy) + 1 = 0 \\ \Longrightarrow\ & xy = \frac{4 \pm \sqrt{12}}{2} \\ & \quad = 2 \pm \sqrt{3}. \end{aligned}$$

So, the graph consists of two reciprocal graphs. Since both roots $2 \pm \sqrt{3}$ are positive, the graph has two branches in the positive quadrant, and two branches in the negative quadrant:



4293. (a) This is a two-tailed test, giving the alternative hypothesis as $H_1 : p \neq 0.34$.

(b) For a sample of 50, under the assumption of H_0, the distribution is $X \sim B(50, 0.34)$. The critical region is split into 2.5% at each tail. Using the binomial facility on a calculator,

$$\begin{aligned} \mathbb{P}(X \leq 10) &= 0.0227 < 0.025, \\ \mathbb{P}(X \leq 11) &= 0.0467 > 0.025. \end{aligned}$$

So, the critical value at the lower tail is 10. At the upper tail,

$$\begin{aligned} \mathbb{P}(X \geq 23) &= 0.0141 < 0.025, \\ \mathbb{P}(X \geq 24) &= 0.0282 > 0.025. \end{aligned}$$

So, the critical value at the upper tail is 24. The critical region is all values including and outside the critical values:

$$\{k \in \mathbb{N} : 0 \leq k \leq 10\} \cup \{k \in \mathbb{N} : 24 \leq k \leq 50\}.$$

(c) The sample statistic $x = 11$ does not lie in the critical region, so there is insufficient evidence, at the 5% level, to reject H_0. It seems that the null hypothesis holds.

(d) The test statistic $x = 11$ does provide evidence against H_0. From only the first sample, this is insufficient: the conclusion is that H_0 is not rejected. But a second sample provides *more* evidence against H_0.

The combined sample does, in fact, provide sufficient evidence to reject H_0. Grouping with $n = 100$ and $x = 22$, the p-value is 0.00621, which is a long way below 0.025.

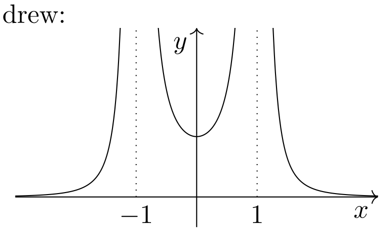
4294. Using the second Pythagorean trig identity,

$$\begin{aligned}&\int_0^{\frac{\pi}{4}} \tan^2 x\,dx\\ &= \int_0^{\frac{\pi}{4}} \sec^2 x - 1\,dx\\ &= \Big[\tan x - x\Big]_0^{\frac{\pi}{4}}\\ &= \big(1-\tfrac{\pi}{4}\big) - \big(0-0\big)\\ &= 1-\tfrac{\pi}{4},\ \text{as required.}\end{aligned}$$

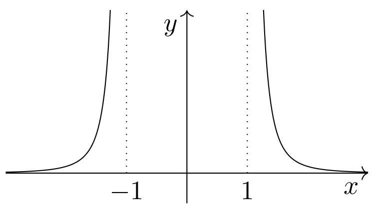
4295. Rearranging and squaring both sides,

$$y = \frac{1}{(x+1)^2(x-1)^2}.$$

This has double asymptotes at $x = \pm 1$, and tends asymptotically to the x axis. Its graph is what the student drew:



But what the student has forgotten is that we have introduced new solution points by squaring. The original curve has no points where $x^2 - 1 < 0$, which is for $-1 < x < 1$. The correct graph is



4296. For period 2, we require $x_{n+2} = x_n$. Setting this equation up,

$$\begin{aligned}&x = \frac{a}{b - \frac{a}{b-x}}\\ \Longrightarrow\ &x = \frac{ab-ax}{b^2-bx-a}\\ \Longrightarrow\ &x(b^2-bx-a) = ab-ax\\ \Longrightarrow\ &bx^2 - b^2x + ab = 0.\end{aligned}$$

If $b = 0$, then $a = \pm 1$ exhibits period 2 behaviour. If $b \neq 0$, we can divide by it, giving $x^2 - bx + a = 0$. For roots, we require $b^2 - 4a \geq 0$, so $b^2 \geq 4a$. $\square$

4297. (a) This is true. The graph is akin to $x^2 + y^2 = 1$.

(b) This is false. However large the value of x, there is always a value of y which will satisfy $x^5 + y^5 = 1$. The graph tends asymptotically to $y = x$.

(c) This is true. The graph is akin to $x^4 + y^4 = 1$.

4298. Since k_1, k_2 are integers, the graph is periodic. It has period $\frac{2\pi}{k}$, where $k = \text{lcm}(k_1, k_2)$. The graph is non-constant and has no discontinuities, so there are at least two SPs in each period, one minimum and one maximum. So, since there are infinitely many periods, there must be infinitely many SPs. QED.

4299. The normal passes through $\big(p, p^2\big)$ with gradient $-\frac{1}{2p}$. So, it has equation

$$y - p^2 = -\tfrac{1}{2p}(x-p).$$

Solving simultaneously with $y = x^2$,

$$\begin{aligned}&x^2 - p^2 = -\tfrac{1}{2p}(x-p)\\ \Longrightarrow\ &2px^2 + x - p - 2p^3 = 0\\ \Longrightarrow\ &x = \frac{-1 \pm \sqrt{1 + 8p^2(1+2p^2)}}{4p}\\ &= \frac{-1 \pm \sqrt{(4p^2+1)^2}}{4p}\\ &= \frac{-1 \pm (4p^2+1)}{4p}\\ &= p, \frac{-2p^2-1}{2p}.\end{aligned}$$

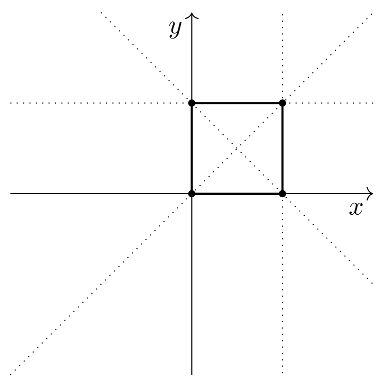
The former is the original point $\big(p, p^2\big)$. The latter is the re-intersection, which occurs at

$$y = \left(\frac{-2p^2-1}{2p}\right)^2 \equiv \frac{(2p^2+1)^2}{4p^2}.$$

4300. Values at which the mod functions switch on are at $y + x - 1 = 0$ and $y - x = 0$. These are a pair of perpendicular lines. Anywhere except for on these lines, the locus of R must consist of straight line segments. The four straight lines segments are

$$\begin{aligned}(y+x-1) + (y-x) = 1 &\implies y = 1,\\ (y+x-1) - (y-x) = 1 &\implies x = 1,\\ -(y+x-1) + (y-x) = 0 &\implies x = 0,\\ -(y+x-1) - (y-x) = 0 &\implies y = 0.\end{aligned}$$

This gives the locus as a unit square:



—— End of 43rd Hundred ——